

ON ISOLATED SINGULARITIES WITH NONINVERTIBLE FINITE ENDOMORPHISM

YUCHEN ZHANG

ABSTRACT. We prove that if $\phi : (X, 0) \rightarrow (X, 0)$ is a finite endomorphism of an isolated singularity such that $\deg(\phi) \geq 2$ and ϕ is étale in codimension 1, then X is \mathbb{Q} -Gorenstein and log canonical.

1. INTRODUCTION

Let us start with an easy example. Let C be a smooth curve of genus g . By an argument using Riemann-Hurwitz Theorem, one can see that C has a finite endomorphism ϕ of degree ≥ 2 if and only if $g \leq 1$. In this case, assume that there is an ample divisor H on C such that ϕ^*H is a multiple of H . Then ϕ induces a finite endomorphism on the cone X over C with polarization mH , where m is a sufficiently large integer. On the other hand, one can see easily by adjunction that, a normal cone over a smooth curve of genus g has log canonical singularity if and only if $g \leq 1$. This phenomena is true in general. In this paper, we prove the following theorem:

Theorem 1.1. *Let $(X, 0)$ be a normal projective variety with isolated singularity $0 \in X$. Suppose that there exists a finite endomorphism $\phi : (X, 0) \rightarrow (X, 0)$ such that $\deg \phi \geq 2$ and ϕ is étale in codimension 1. Then X is \mathbb{Q} -Gorenstein and log canonical.*

It is necessary to assume that X has isolated singularity. Otherwise, let $X = E \times V$, where E is an elliptic curve and V is an arbitrary variety with bad singularity. Then X has an induced noninvertible étale endomorphism from E .

We briefly review the history of this problem. For the definitions of related terminologies, we refer to Section 2 and [BdFF12]. The surface case is studied in [Wahl90]. Let X be a normal surface and $f : Y \rightarrow X$ be the minimal resolution. The relative Zariski decomposition yields $K_{Y/X} = P + N$. Wahl's invariant is defined as the nonnegative intersection number $-P^2$, which is the key ingredient in the study of surfaces with noninvertible finite endomorphisms. A classification of such surfaces is given in [Favre10, FN05]. Wahl's invariant is generalized to higher

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dimensions by Boucksom, de Fernex and Favre [BdFF12]. Due to the absence of minimal resolution, they consider log discrepancy divisors on all birational models over X as Shokurov's b -divisor $A_{\mathcal{X}/X}$. Zariski decomposition is replaced by the nef envelope $\text{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$. It can be shown that $-(\text{Env}_{\mathcal{X}}(A_{\mathcal{X}/X}))^n$ is a well-defined finite nonnegative number, which is called $\text{vol}_{\text{BdFF}}(X)$. This volume behaves well under finite morphisms. In particular, the following theorem is proved:

Theorem 1.2. [BdFF12, Theorem A and B] *For normal isolated singularities $(X, 0)$ with noninvertible finite endomorphism, $\text{vol}_{\text{BdFF}}(X) = 0$. Moreover, when X is \mathbb{Q} -Gorenstein, $\text{vol}_{\text{BdFF}}(X) = 0$ if and only if X has log canonical singularity.*

The same theorem is obtained in [BH14] by analyzing the behavior of non-log-canonical centers under finite pullback.

Unfortunately, in [yZhang14], the author produces a non- \mathbb{Q} -Gorenstein isolated singularity $(X, 0)$ such that $\text{vol}_{\text{BdFF}}(X) = 0$ while there is no boundary Δ such that (X, Δ) is log canonical. We should remark that in this example, X admits a small log canonical modification [OX12][BH14, Proposition 2.4].

The \mathbb{Q} -Gorenstein case is further studied in [yZhang14, Section 3]. Specifically, the author shows that, like the surface case, $\text{vol}_{\text{BdFF}}(X)$ can be calculated by an intersection number on a certain birational model $f : Y \rightarrow X$, namely, the log canonical modification [OX12]. A key property of such model is that $K_Y + E_f$ is f -ample, where E_f is the reduced exceptional divisor. In the non- \mathbb{Q} -Gorenstein case, the existence of the log canonical modification is conjectured to be true following from full minimal model program including abundance, but has not yet been proved.

In this paper, in order to show that a normal isolated singularity $(X, 0)$ in Theorem 1.1 is indeed \mathbb{Q} -Gorenstein, we consider a birational model over X called movable modification (Theorem 3.1) where $K_Y + E_f$ is f -movable. The existence of movable modification is known to experts. However, we include a proof in Section 3. We introduce a new volume $\text{vol}_{\text{mov}}(X)$ (Definition 4.6) using Nakayama's σ -decomposition. We show the following theorem:

Theorem 1.3 (Proposition 3.5, Lemma 4.7, Theorem 4.8, 5.3 and 5.4). *For normal isolated singularities $(X, 0)$ and $(Y, 0)$,*

- (1) *$\text{vol}_{\text{mov}}(X)$ is a finite nonnegative number, $\text{vol}_{\text{mov}}(X) \geq \text{vol}_{\text{BdFF}}(X)$ with equation when X is \mathbb{Q} -Gorenstein.*
- (2) *If $\phi : (Y, 0) \rightarrow (X, 0)$ is a finite morphism of degree d that is étale in codimension 1, then*

$$\text{vol}_{\text{mov}}(Y) = d \text{vol}_{\text{mov}}(X).$$

- (3) *If $\phi : (X, 0) \rightarrow (X, 0)$ is a finite endomorphism of degree ≥ 2 that is étale in codimension 1, then*

$$\text{vol}_{\text{mov}}(X) = 0.$$

- (4) If $\text{vol}_{\text{mov}}(X) = 0$, then X is numerically \mathbb{Q} -Gorenstein.
- (5) Every numerically log canonical variety is \mathbb{Q} -Gorenstein.

As a byproduct while studying numerically \mathbb{Q} -Gorenstein varieties, we obtain the following theorem which slightly generalizes [OX12, Theorem 1.1].

Theorem 1.4 (Theorem 3.6). *Let X be a numerically \mathbb{Q} -Gorenstein projective variety. Then there exists a log canonical model Y over X .*

The case that ϕ is not étale in codimension one is also interesting. Although X is not \mathbb{Q} -Gorenstein in general (see example below), we still expect that there exists a boundary Δ such that (X, Δ) is log canonical. It is known that X has klt singularities assuming that X is \mathbb{Q} -Gorenstein [BdFF12, Theorem B].

Example 1.5. Let $V = \mathbb{P}^1 \times E$ where E is an elliptic curve. Let $\phi_1 : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be raising to the fourth power and $\phi_2 = [2] : E \rightarrow E$ be multiplying by 2. Let H_1 be a point in \mathbb{P}^1 , H_2 be a symmetric ample divisor on E (i.e., $[-1]^*H_2 = H_2$) and $H = p_1^*H_1 + p_2^*H_2$, where p_i are the projections. Then $\phi = \phi_1 \times \phi_2$ is an endomorphism on V of degree 16 such that $\phi^*H \sim 4H$. We may take X be the cone over V with polarization H . Thus, ϕ induces an endomorphism on X of degree 16. But X is not \mathbb{Q} -Gorenstein or log terminal. However, one may take Δ be the cone over $p_1^*(D_1 + D_2)$ for two different points D_1 and D_2 on \mathbb{P}^1 and see that $K_X + \Delta$ is \mathbb{Q} -Cartier and log canonical.

Question 1.6. *Let $(X, 0)$ be a normal isolated singularity and (V, H) be a normal polarized projective variety. If $\phi : (X, 0) \rightarrow (X, 0)$ is a noninvertible finite endomorphism that is not étale in codimension one, does there exist a boundary Δ such that (X, Δ) is log canonical?*

The global counterpart of this problem is well studied. Let (V, H) be a normal polarized projective variety. A finite endomorphism $\phi : V \rightarrow V$ is called polarized if ϕ^*H is a multiple of H . The cone over a smooth variety V with the polarization H rises an isolated singularity as in the local case. The classification of polarized Kähler surfaces (also known as dynamic surfaces) is obtained in [FN05] and [sZhang06, Proposition 2.3.1]. The three dimensional case is studied in [Fujimoto02] and [FN07], where the classification of smooth projective 3-fold with $\kappa(X) \geq 0$ admitting nontrivial endomorphism (which is implied to be étale) is given. Higher dimensions are studied in [NZ09, NZ10][GKP16, Theorem 1.21]. It is known that a \mathbb{Q} -Gorenstein polarized projective variety with noninvertible endomorphism is log canonical [BH14].

There are also conditions that are weaker than being polarized rising from dynamic systems, such as: amplified and unity-free. We refer to [KR15] for the definitions and comparisons.

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2. PRELIMINARIES

Throughout this paper, we work over an algebraically closed field k of characteristic 0.

2.1. Movable divisors. Let $f : Y \rightarrow X$ be a projective morphism between normal varieties and D be an f -big Cartier divisors on Y . The base locus of D over X , $\text{Bs}(D)$, is the co-support of the image of the following canonical map:

$$f^* f_* \mathcal{O}_Y(D) \otimes \mathcal{O}_Y(-D) \rightarrow \mathcal{O}_Y.$$

The stable base locus is defined to be

$$\mathbf{B}(D) = \bigcap_{m \geq 1} \text{Bs}(mD)_{\text{red}}.$$

One can easily extend this definition to \mathbb{Q} -Cartier divisors. We define the diminished base locus¹ of D as

$$\mathbf{B}_-(D) = \bigcup_{\epsilon > 0} \mathbf{B}(D + \epsilon H),$$

where H is an f -ample divisor on Y . It can be shown that $\mathbf{B}_-(D)$ is independent of the choice of H (see [ELMNP06, Section 1]).

We say an f -pseudo-effective \mathbb{Q} -Cartier divisor D on Y is f -mobile if

$$\text{codim}(\text{Bs}(D)) \geq 2.$$

The f -**movable cone** $\overline{\text{Mov}}(Y/X)$ is the closure of the cone generated by f -mobile Cartier divisors in $N^1(Y/X)$. We call an \mathbb{R} -divisor D f -movable if $D \in \overline{\text{Mov}}(Y/X)$.

Lemma 2.1. [Nakayama04, Theorem V.1.3] *Let D be a \mathbb{Q} -Cartier divisor on Y . Then $D \in \overline{\text{Mov}}(Y/X)$ if and only if $\mathbf{B}_-(D)$ contains no divisor.* \square

Lemma 2.2. *Let $f : Y \rightarrow X$ and $g : Z \rightarrow Y$ be two projective morphisms and $\phi = f \circ g$. If a Cartier divisor D on Z is ϕ -mobile, then D is also g -mobile. In particular, $\overline{\text{Mov}}(Z/X) \subseteq \overline{\text{Mov}}(Z/Y)$.*

Proof. Since the morphism

$$g^* f^* f_* g_* \mathcal{O}_Z(D) \otimes \mathcal{O}_Z(-D) = \phi^* \phi_* \mathcal{O}_Z(D) \otimes \mathcal{O}_Z(-D) \xrightarrow{\rho_\phi} \mathcal{O}_Z$$

factors through

$$g^* g_* \mathcal{O}_Z(D) \otimes \mathcal{O}_Z(-D) \xrightarrow{\rho_g} \mathcal{O}_Z,$$

¹This is called restricted base locus in [ELMNP06] and non-nef locus in [BDPP13]

it follows that $\text{im } \rho_\phi \subseteq \text{im } \rho_g$. Thus, if the co-support of $\text{im } \rho_\phi$ has codimension ≥ 2 , then so is the co-support of $\text{im } \rho_g$. The lemma follows. \square

The following lemma is a generalization of the well-known Negativity Lemma [KM98, Lemma 3.39].

Lemma 2.3. [Fujino11, Lemma 4.2] *Let $f : Y \rightarrow X$ be a proper birational morphism where Y is a normal \mathbb{Q} -factorial variety. Let E be an \mathbb{R} -divisor on Y such that E is f -exceptional and $E \in \overline{\text{Mov}}(Y/X)$. Then $E \leq 0$.* \square

2.2. Shokurov's b -divisor. Let X be a normal variety.

A **Weil b -divisor** \mathcal{W} over X is to assign every birational model $f : Y \rightarrow X$ a Weil divisor \mathcal{W}_Y on Y that is compatible with pushforwards: if $\phi : Z \rightarrow Y$ are two models over X , then $\phi_*(\mathcal{W}_Z) = \mathcal{W}_Y$. The Weil-divisor \mathcal{W}_Y is called the **trace** of \mathcal{W} on Y . We denote by $\text{Div}(\mathcal{X})$ the group of Weil b -divisors over X and define \mathbb{Q} -Weil (\mathbb{R} -Weil, resp.) b -divisor as elements of $\text{Div}(\mathcal{X}) \otimes \mathbb{Q}$ ($\text{Div}(\mathcal{X}) \otimes \mathbb{R}$, resp.).

We call the Weil b -divisor \mathcal{C} a **Cartier b -divisor** over X if there exists a birational model $f : Y \rightarrow X$ such that \mathcal{C}_Y is Cartier and for every other model $\phi : Z \dashrightarrow Y$ with common resolution

$$\begin{array}{ccc} & W & \\ s \swarrow & & \searrow t \\ Z & \text{-----} & Y \end{array}$$

we have $\mathcal{C}_Z = s_*(t^*\mathcal{C}_Y)$. In this case, we say that $f : Y \rightarrow X$ is a **determinant** of \mathcal{C} . Similarly, we define \mathbb{Q} -Cartier b -divisor and \mathbb{R} -Cartier b -divisor.

A \mathbb{R} -Cartier b -divisor \mathcal{C} over X is called **X -nef** if there exists a (hence any) determinant $f : Y \rightarrow X$ such that \mathcal{C}_Y is f -nef. We call a \mathbb{R} -Weil b -divisor \mathcal{W} X -nef if \mathcal{W} is a limit of X -nef \mathbb{R} -Cartier b -divisors, where the limit is taken in the numerical class of every model. The following lemma gives a characterization of X -nef \mathbb{R} -Weil b -divisors.

Lemma 2.4. [BdFF12, Lemma 2.10] *A \mathbb{R} -Weil b -divisor \mathcal{W} is X -nef if and only if $\mathcal{W}_Y \in \overline{\text{Mov}}(Y/X)$ on every \mathbb{Q} -factorial model Y .* \square

Let \mathcal{W}_1 and \mathcal{W}_2 be two \mathbb{R} -Weil b -divisors. We say $\mathcal{W}_1 \geq \mathcal{W}_2$, if for every model Y , we have $(\mathcal{W}_1)_Y \geq (\mathcal{W}_2)_Y$.

2.3. Nakayama's σ -decomposition. Let X be a \mathbb{Q} -factorial projective normal variety over S and D be an S -big \mathbb{R} -divisor on X . For every prime divisor Γ on X , we define

$$\sigma_\Gamma(D) = \inf\{\text{mult}_\Gamma \Delta \mid \Delta \equiv_S D, \Delta \geq 0\}.$$

Since D is S -big, there exists an effective divisor Δ that is numerically equivalent to D . Hence, the infimum above is not taken over an empty set.

Remark 2.5. By the observation of Nakayama, this definition can be generalized to S -pseudo-effective \mathbb{R} -divisors. Let D be an S -pseudo-effective divisor on X . Fix an S -ample divisor A on X . Since for every $\epsilon > 0$, $D + \epsilon A$ is S -big, we can define $\sigma_\Gamma(D) = \lim_{\epsilon \downarrow 0} \sigma_\Gamma(D + \epsilon A)$. It is shown in [Nakayama04, Lemma III.1.4-5] that $\sigma_\Gamma(D)$ is independent of the choice of A and only depends on the numerical class of D . However, an example is given in [Lesieutre15] that this limit can be ∞ when S is not a point. In this paper, we only use the σ -decomposition in the case that X is birational to S , hence every divisor is S -big (see [Nakayama04, Lemma III.1.4(2)]).

It is shown in [Nakayama04, Lemma III.4.2] that there are finitely many prime divisors Γ on X such that $\sigma_\Gamma(D) > 0$, which leads us to the following definition.

Definition 2.6. Let D be an S -big \mathbb{R} -divisor on a \mathbb{Q} -factorial projective variety X over S . We define

$$N_\sigma(D) = \sum \sigma_\Gamma(D) \Gamma \quad \text{and} \quad P_\sigma(D) = D - N_\sigma(D),$$

where $P_\sigma(D)$ and $N_\sigma(D)$ are called the positive and negative part of the σ -decomposition, respectively.

Notation 2.7. To be precise, the σ -decomposition here should be written as $P_{\sigma/S}$ and $N_{\sigma/S}$ as we are using the relative version. However, in order to make our notation concise, we will omit the base S when it is clear from the context.

We record some properties of the σ -decomposition as below.

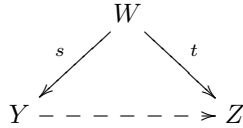
Proposition 2.8. *Let D be an S -big \mathbb{R} -divisor on a \mathbb{Q} -factorial variety X . Let $f : Y \rightarrow X$ be a proper birational morphism, where Y is normal.*

- (1) $N_\sigma(D) = 0$ if and only if $D \in \overline{\text{Mov}}(X/S)$.
- (2) $P_\sigma(D)$ is the largest \mathbb{R} -divisor in $\overline{\text{Mov}}(X/S)$ that is no greater than D .
- (3) $f_* P_\sigma(f^* D) = P_\sigma(D)$. In the other word, $P_\sigma(f^* D)$ defines a \mathbb{R} -Weil b -divisor over X .
- (4) σ_Γ is a continuous function on the big cone of X over S . So is P_σ , where the convergence is coefficiently-wise.

Proof. (1) and (2) are [Nakayama04, Proposition III.1.14]. (3) is [Nakayama04, Theorem III.2.5(1)]. (4) is [Nakayama04, Lemma III.1.7(1)]. \square

Inspired by the above proposition, we give the definition of σ -closure.

Definition 2.9. Let Y be a \mathbb{Q} -factorial model over X and D be an S -big \mathbb{R} -divisor on Y . The σ -closure $\mathcal{P}_\sigma(D)$ is such an \mathbb{R} -Weil b -divisor: for every model Z over X , let W be a model dominating Y and Z .



Then the trace of $\mathcal{P}_\sigma(D)$ on Z is $t_*P_\sigma(s^*D)$. It is well-defined by Proposition 2.8(3).

Lemma 2.10. *The σ -closure is X -nef.*

Proof. The lemma follows directly from Proposition 2.8(2) and Lemma 2.4. \square

2.4. Pullback of Weil divisors. We recall the definition of pullback of Weil divisors from [dFH09, BdFF12] as below.

Let X be a normal variety, D be a \mathbb{R} -Weil divisor on X and E be a prime divisor over X . We define the valuation $v_E(D) = v_E(\mathcal{O}_X(-D)) = v_E(\mathcal{O}_X(\lfloor -D \rfloor))$ as

$$v_E(D) = \min\{v_E(\phi) \mid \phi \in \mathcal{O}_U(\lfloor -D \rfloor), U \cap c_X(E) \neq \emptyset\}.$$

If D is Cartier, we see that $v_E(D)$ is the usual valuation.

Suppose $f : Y \rightarrow X$ is a proper birational morphism and Y is normal. The natural pullback $f^\natural D$ is defined as

$$f^\natural D = \sum v_E(D)E,$$

where the sum runs through all prime divisors on Y . It is easy to see that $\mathcal{O}_Y(-f^\natural D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{\vee\vee}$. The natural pullback is also known as $Z(\mathcal{O}_X(-D))_f$ in [BdFF12], where $Z(\mathcal{O}_X(-D))$ is viewed as a Weil b -divisor consisting of natural pullbacks.

In general, the natural pullbacks do not behave well under composition.

Lemma 2.11. [dFH09, Lemma 2.7] *Let $f : Y \rightarrow X$ and $g : Z \rightarrow Y$ be two proper birational morphisms between normal varieties, D be an \mathbb{R} -Weil divisor on X . Then $(f \circ g)^\natural D - g^\natural(f^\natural D)$ is not necessarily zero. However, it is effective and g -exceptional. Moreover, if $\mathcal{O}_X(-D) \cdot \mathcal{O}_Y$ is an invertible sheaf on Y , then $(f \circ g)^\natural D = g^\natural(f^\natural D)$. \square*

Lemma 2.12. [BdFF12, Lemma 1.8 and 2.10]

- (1) *If $\mathcal{O}_X(-D) \cdot \mathcal{O}_Y$ is an invertible sheaf, then $-f^\natural D$ is relatively globally generated over X . In the other word, the following canonical homomorphism is surjective.*

$$f^* f_* \mathcal{O}_Y(-f^\natural D) \rightarrow \mathcal{O}_Y(-f^\natural D).$$

- (2) *If $-f^\natural D$ is \mathbb{Q} -Cartier, then $-f^\natural D \in \overline{\text{Mov}}(Y/X)$.*

Proof. (1) Since $\mathcal{O}_X(-D) \cdot \mathcal{O}_Y$ is the image of $f^* \mathcal{O}_X(-D)$ under the homomorphism $f^* \mathcal{O}_X(-D) \rightarrow f^* \mathcal{O}_X$, we have a surjection

$$f^* \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X(-D) \cdot \mathcal{O}_Y.$$

It is obvious that $f_* \mathcal{O}_Y(-f^\natural D) = \mathcal{O}_X(-D)$. The statement follows.

- (2) Suppose $-mf^\natural D$ is Cartier. Let $g : Z \rightarrow X$ be a log resolution of X such that $\mathcal{O}_X(-D) \cdot \mathcal{O}_Z$ is an invertible sheaf and g factors through f via $\phi : Z \rightarrow Y$. Such log resolution exists by [dFH09, Theorem 4.2]. By part

(1), $-mg^{\natural}D$ is relatively globally generated over X . Hence, $-mf^{\natural}D = \phi_*(-mg^{\natural}D)$ is f -mobile. \square

It is shown in [dFH09, Lemma 2.8] that for every positive integer m , we have $f^{\natural}D \geq \frac{1}{m}f^{\natural}(mD)$. Thus, we can define the pullback of D as a \mathbb{R} -Weil divisor,

$$f^*D = \liminf_{m \rightarrow \infty} \frac{f^{\natural}(mD)}{m}.$$

It is not hard to see that the \liminf above is actually a limit ([BdFF12, Lemma 2.1]).

Lemma 2.13. *If Y is \mathbb{Q} -factorial, then $-f^*D \in \overline{\text{Mov}}(Y/X)$.*

Proof. This is obvious by Lemma 2.12 and the definition. \square

Lemma 2.14. *Let $f : Y \rightarrow X$ and $g : Z \rightarrow Y$ be two proper birational morphisms between normal varieties. Then for every \mathbb{Q} -Weil divisor D on X , $(f \circ g)^*D - g^*(f^*D)$ is an effective g -exceptional divisor.*

Proof. The lemma follows from Lemma 2.11 and definition. \square

We define $\text{Env}_X(D)$ to be the \mathbb{R} -Weil b -divisor whose trace on every model Y is $-f^*(-D)$ (see [BdFF12, Remark 2.4]). It is shown in [BdFF12, Corollary 2.13] that $\text{Env}_X(D)$ is the largest X -nef \mathbb{R} -Weil b -divisor \mathcal{W} such that $\mathcal{W}_X = D$.

Lemma 2.15. *In the setting of Lemma 2.14, if Z is \mathbb{Q} -factorial, then $-(f \circ g)^*D = P_{\sigma}(-g^*(f^*D))$. Hence, $\text{Env}_X(-D) = P_{\sigma}(-f^*D)$.*

Proof. By Lemma 2.13, $-(f \circ g)^*D$ is a movable \mathbb{R} -divisor that is no greater than $-g^*(f^*D)$. Using Lemma 2.8, we have $-(f \circ g)^*D \leq P_{\sigma}(-g^*(f^*D))$. On the other hand, $P_{\sigma}(-f^*D)$ is an X -nef \mathbb{R} -Weil b -divisor whose trace on X is D . Hence, $\text{Env}_X(-D) \geq P_{\sigma}(-f^*D)$. The lemma follows. \square

Lemma 2.16. [BdFF12, Corollary 2.13] *If $\text{Env}_X(D)$ is \mathbb{R} -Cartier with a determinant on Y , then the trace of $\text{Env}_X(D)$ on Y is X -nef.* \square

We record the following Negativity Lemma in the context of b -divisors.

Lemma 2.17. [BdFF12, Proposition 2.12] *Let \mathcal{W} be an X -nef \mathbb{R} -Weil b -divisor over X and $f : Y \rightarrow X$ be a birational model. Then $\mathcal{W}_Y \leq -f^*(-\mathcal{W}_X)$.* \square

2.5. Nef envelope of \mathbb{R} -Weil b -divisors and volume. All the definitions below are from [BdFF12], where the reader can find the details.

Let \mathcal{W} be an \mathbb{R} -Weil b -divisor over X . We define the nef envelope $\text{Env}_{\mathcal{X}}(\mathcal{W})$ of \mathcal{W} to be the largest X -nef \mathbb{R} -Weil b -divisor \mathcal{Z} such that $\mathcal{Z} \leq \mathcal{W}$, if exists. Let \mathcal{C} be an \mathbb{R} -Cartier b -divisor over X . The self-intersection $\mathcal{C}^n = \mathcal{C}_Y^n$ if Y is a determinant of \mathcal{C} . It is obvious that \mathcal{C}^n is independent of the choice of Y . If \mathcal{W} is X -nef whose

center on X is $0 \in X$, we define $\mathcal{W}^n = \inf\{\mathcal{C}^n\}$ where the infimum is taken for all X -nef \mathbb{R} -Carter b -divisors such that $\mathcal{C} \geq \mathcal{W}$.

We copy the following properties from [BdFF12].

Proposition 2.18. [BdFF12, Theorem 4.14 and Proposition 4.16] *Let \mathcal{W} be an X -nef \mathbb{R} -Weil b -divisor whose center on X is $0 \in X$.*

- (1) *If $\mathcal{W}_1 \leq \mathcal{W}_2$ are two X -nef \mathbb{R} -Weil b -divisors over X , then $\mathcal{W}_1^n \leq \mathcal{W}_2^n \leq 0$.*
- (2) *$\mathcal{W}^n = 0$ if and only if $\mathcal{W} = 0$.*
- (3) *Let $\phi : (Y, 0) \rightarrow (X, 0)$ be a finite map of degree d between isolated singularities. Then $(\phi^*\mathcal{W})^n = d(\mathcal{W}^n)$.* \square

For a model $f : Y \rightarrow X$, we fix canonical divisors such that $f_*K_Y = K_X$. For every positive integer m , we define the m -th **limiting log discrepancy divisor** as

$$A_{m,Y/X} = K_Y + E_f - \frac{1}{m}f^*(mK_X),$$

where E_f is the reduced exceptional divisor. We denote by $A_{m,\mathcal{X}/X}$ the corresponding \mathbb{Q} -Weil b -divisor. Similarly, we define the **log discrepancy divisor** as

$$A_{Y/X} = K_Y + E - f^*K_X$$

and $A_{\mathcal{X}/X}$ to be the corresponding \mathbb{R} -Weil b -divisor.

It is shown in [BdFF12, yZhang14] that if X has isolated singularity, then $\text{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$ and $\text{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$ exist. We define

$$\text{vol}_m(X) = -\text{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})^n \quad \text{and} \quad \text{vol}_{\text{BdFF}}(X) = -\text{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})^n.$$

It is known that these volumes are finite nonnegative numbers.

In the case that X is \mathbb{Q} -Gorenstein, it is shown in [yZhang14, Theorem 3.2] that $\text{vol}_{\text{BdFF}}(X) = \text{vol}_m(X) = -A_{Y/X}^n$ for m sufficiently large and divisible, where Y is the log canonical modification of X . When X is non- \mathbb{Q} -Gorenstein, [yZhang14, Corollary 4.3] shows that for every $m \geq 2$, one can always pick a boundary Δ on X such that $\text{vol}_m(X)$ is calculated on the log canonical modification of the pair (X, Δ) . We do not know whether $\text{vol}_{\text{BdFF}}(X)$ can be calculated by intersection number on some model.

The following theorem gives a criteria for log canonical singularity.

Theorem 2.19. [yZhang14, Corollary 4.6] *If $\text{vol}_m(X) = 0$ for some (hence any multiple of) integer $m \geq 1$, then there exists a boundary Δ on X such that (X, Δ) is log canonical.* \square

2.6. Numerically \mathbb{Q} -Cartier divisors. The numerically \mathbb{Q} -Cartier divisors are defined in [BdFF12] and are further studied in [BdFFU14], which behave like \mathbb{Q} -Cartier divisors under birational pullbacks.

Definition 2.20. Let X be a normal variety.

- (1) A Weil divisor D on X is numerically Cartier if there exists a resolution of singularities $f : Y \rightarrow X$ and an f -numerically trivial Cartier divisor D' on Y such that $f_*D' = D$.
- (2) A \mathbb{Q} -Weil divisor is numerically \mathbb{Q} -Cartier if some multiple is numerically Cartier.
- (3) When the canonical divisor K_X is numerically \mathbb{Q} -Cartier, we say X is numerically \mathbb{Q} -Gorenstein.

Proposition 2.21. *Let D be a Weil divisor on X .*

- (1) *D is numerically \mathbb{Q} -Cartier if and only if $\text{Env}_X(-D) = -\text{Env}_X(D)$.*
- (2) *Suppose X has rational singularities. Then D is numerically \mathbb{Q} -Cartier if and only if D is \mathbb{Q} -Cartier.*
- (3) *Suppose D is numerically \mathbb{Q} -Cartier. Let $f : Y \rightarrow X$ be a birational model such that f^*D is \mathbb{Q} -Cartier. Then f^*D is f -numerically trivial and $-\text{Env}_X(-D) = \overline{f^*D}$.*

Proof. (1) is [BdFFU14, Proposition 5.9] and (2) is [BdFFU14, Theorem 5.11].

For (3), let $g : Z \rightarrow Y$ be any \mathbb{Q} -factorial birational model over Y and $\phi = f \circ g$. By Lemma 2.15, we have

$$-\phi^*D = P_\sigma(-g^*(f^*D)) \leq -g^*(f^*D),$$

and

$$-\phi^*(-D) = P_\sigma(-g^*(f^*(-D))) \leq -g^*(f^*(-D)).$$

Applying (1), the sum yields

$$0 = -\phi^*D - \phi^*(-D) \leq -g^*(f^*D) - g^*(f^*(-D)) = 0.$$

Hence, $\text{Env}_X(D) = \overline{f^*D}$ and $-\text{Env}_X(D) = \overline{-f^*D}$. By Lemma 2.16, both f^*D and $-f^*D$ are f -nef. Thus, f^*D is f -numerically trivial. \square

3. PARTIAL RESOLUTIONS

3.1. Movable modification of non- \mathbb{Q} -Gorenstein varieties. The construction of our volume depends on the following theorem which is known to experts.

Theorem 3.1. *There exists a birational morphism $f : Y \rightarrow X$ such that*

- (1) *Y is \mathbb{Q} -factorial,*
- (2) *(Y, E_f) is dlt, and*
- (3) *$K_Y + E_f \in \overline{\text{Mov}}(Y/X)$,*

where E_f is the reduced exceptional divisor. Moreover, $A_{Y/X} \leq 0$. We call $f : Y \rightarrow X$ a movable modification of X .

Remark 3.2. Theorem 3.1 does not require that X is \mathbb{Q} -Gorenstein. It is proved in [OX12] that in the setting of $K_X + \Delta$ being \mathbb{Q} -Cartier, one may further require that $K_Y + E_f + \tilde{\Delta}$ is f -semi-ample, where $\tilde{\Delta}$ is the strict transform of Δ on Y .

The general case of semi-ampleness is true if one assume the full minimal model program (MMP) including abundance.

Proof. Let $g : Z \rightarrow X$ be a log resolution and E_g be the reduced exceptional divisor. Let H be a g -ample divisor on Z . We run the $(K_Z + E_g)$ -MMP with scaling H over X . We obtain a sequence of flips and divisorial contractions:

$$Z = Z_0 \dashrightarrow Z_1 \dashrightarrow \cdots,$$

and a decreasing sequence

$$\lambda_i = \inf\{s \in \mathbb{R} \mid K_{Y_i} + \Delta_i + tH_i \text{ is nef over } X\}.$$

We know that for every $\lambda_i \geq t \geq \lambda_{i+1}$, $K_{Z_i} + E_{g_i} + tH_i$ is relatively semi-ample over X and $\lim_{i \rightarrow \infty} \lambda_i = 0$ (see [OX12, Lemma 2.6]).

For every divisor $E \subset \mathbf{B}_-(K_Z + E_g/X)$ or equivalently every component E of $N_\sigma(K_Z + E_g)$, there is some $t > 0$ such that $E \subset \mathbf{B}(K_Z + E_g + tH/X)$. We may find an i such that $\lambda_i \geq t \geq \lambda_{i+1}$. Since $K_{Z_i} + E_{g_i} + tH_i$ is relatively semi-ample over X , E must be contracted on Z_i . Since there are finitely many such E , we conclude that there is a model Z_j such that $\mathbf{B}_-(K_{Z_j} + E_{g_j}/X)$ contains no divisor.

By Lemma 2.1, we have $K_{Z_j} + E_{g_j} \in \overline{\text{Mov}}(Z_j/X)$. The theorem follows by setting $Y = Z_j$.

For the moreover part, by Lemma 2.13, $-f^*K_X \in \overline{\text{Mov}}(Y/X)$, hence so is $K_Y + E_f - f^*K_X$. By Lemma 2.3, we have $A_{Y/X} = K_Y + E_f - f^*K_X \leq 0$. \square

3.2. Log canonical modification of numerically \mathbb{Q} -Gorenstein varieties.

Definition 3.3. For a numerically \mathbb{Q} -Gorenstein variety X , we say X is numerically log canonical (numerically log terminal, resp.) if for every exceptional divisor E on $\phi : Z \rightarrow X$, $\text{ord}_E(K_Z - f^*K_X) \geq -1$ (> -1 , resp.).

Remark 3.4. The above definition coincide with Mumford's numerically pullbacks [KM98, Notation 4.1] on surfaces.

The following proposition generalizes the well-known property in the surface case [KM98, Notation 4.1][dFH09, Proposition 7.14].

Proposition 3.5. *If X is numerically log canonical, then X is \mathbb{Q} -Gorenstein.*

Proof. Let $f : Y \rightarrow X$ be a movable modification of X . Then $K_Y + E_f - f^*K_X \leq 0$ by Lemma 2.3. Since X is numerically log canonical, there must be $K_Y + E_f = f^*K_X$. Thus, (Y, E_f) is a dlt pair such that $K_Y + E_f$ is f -numerically trivial. By the abundance theorem [FG14, Theorem 4.9], $K_Y + E_f$ is f -semi-ample. Let $\phi : Z = \mathbf{Proj}_X \bigoplus_m f_* \mathcal{O}_Y(m(K_Y + E_f)) \rightarrow X$ be the log canonical modification over X . Then $K_Z + E_\phi$ is ϕ -ample. On the other hand, $K_Z + E_\phi = \phi^*K_X$ is ϕ -numerically trivial by Proposition 2.21(3). We conclude that ϕ is an isomorphism. In particular, X is \mathbb{Q} -Gorenstein. \square

The existence of log canonical modification is predicted by full relative minimal model program and is proved for a pair $(X, \Delta = \sum a_i \Delta_i)$ such that $K_X + \Delta$ is \mathbb{Q} -Cartier and $a_i \in [0, 1]$ [OX12]. The same idea applies in the case that $K_X + \Delta$ is numerically \mathbb{Q} -Cartier. For the reader's convenience, we include a sketch below for the case $\Delta = 0$.

Theorem 3.6. *Let X be a numerically \mathbb{Q} -Gorenstein normal variety. Then there exists a unique birational model $f : Y \rightarrow X$, such that*

- (1) $K_Y + E_f$ is a f -ample \mathbb{Q} -Cartier divisor, and
- (2) (Y, E_f) is log canonical,

where E_f is the reduced exceptional divisor.

Proof. Step 1. By Theorem 3.1, there exists a movable modification $\phi : Z \rightarrow X$ such that Z is \mathbb{Q} -factorial, $K_Z + E_\phi \in \overline{\text{Mov}}(Z/X)$ and (Z, E_ϕ) is dlt. By Lemma 2.1, we know that $\mathbf{B}_-(K_Z + E_\phi/X)$ contains no divisor. We denote the complement of the support of $\phi(-\lfloor A_{Z/X} \rfloor)$ by X_{lc} .

Step 2. We show that if E is an exceptional divisor on Z such that $\phi(E) \subseteq X \setminus X_{lc}$, then $\text{ord}_E(A_{Z/X}) < 0$. If this is not true, by Kollár-Shokurov's Connectedness Lemma [KM98, Corollary 5.49], there exist exceptional divisors E_0 and E_1 with centers in $X \setminus X_{lc}$ such that $\text{ord}_{E_0}(A_{Z/X}) = 0$, $\text{ord}_{E_1}(A_{Z/X}) < 0$ and $E_0 \cap E_1 \neq \emptyset$. Then $(A_{Z/X} + \epsilon H)|_{E_0}$ is not effective for $0 < \epsilon \ll 1$ and H ample. Since ϕ^*X is numerically trivial by Proposition 2.21(3), we have $E_0 \subseteq \mathbf{B}_-(K_Z + E_\phi/X)$, a contradiction.

Step 3. Set $B = E_\phi - \epsilon A_{Z/X}$ for some small ϵ such that $B \geq 0$. We show that (Z, B) has a good minimal model over X . The idea is to apply [HX13, Theorem 1.1] over the open subset X_{lc} . A good minimal model exists over X_{lc} by Proposition 3.5 and [HX13, Lemma 2.11]. One can check that no strata of $\lfloor B \rfloor$ is contained in $\phi^{-1}(X \setminus X_{lc})$ using Step 2.

Step 4. Since $K_Z + B = (1 + \epsilon)(K_Z + E_\phi) - \epsilon \phi^*K_X \equiv_f (1 + \epsilon)(K_Z + E_\phi)$, a good minimal model of $K_Z + B$ is also a good minimal model of $K_Z + E_\phi$. Hence the canonical ring $R = \bigoplus_m \phi_* \mathcal{O}_Z(m(K_Z + E_\phi))$ is finitely generated over X and we may take $Y = \mathbf{Proj}_X R$. \square

4. MOVABLE VOLUME

We start with a lemma showing the behavior of Zariski decomposition in the sense of [BdFF12] on a movable modification.

Lemma 4.1. *Let X be a normal variety which is not necessarily \mathbb{Q} -Gorenstein. Let $f : Y \rightarrow X$ be a movable modification. Then the trace of $\text{Env}_X(A_{X/X})$ on Y is $A_{Y/X}$.*

Proof. Consider the \mathbb{R} -Weil b -divisor $\mathcal{P}_- = \mathcal{P}_\sigma(K_Y + E_f) + \text{Env}_X(K_X)$. Let $g : Z \rightarrow Y$ be a projective birational morphism and $\phi = g \circ f$. Since (Y, E_f) is dlt, we

have $g^*(K_Y + E_f) \leq K_Z + E_\phi$. Hence,

$$(\mathcal{P}_-)_Z = \mathcal{P}_\sigma(g^*(K_Y + E_f)) - \phi^*K_X \leq g^*(K_Y + E_f) - \phi^*K_X \leq A_{Z/X}.$$

Thus, \mathcal{P}_- is an X -nef \mathbb{R} -Weil b -divisor that is less than or equal to $A_{X/X}$. By the definition of nef envelope, we have $\mathcal{P}_- \leq \text{Env}_X(A_{X/X})$. Taking the trace on Y , we obtain

$$A_{Y/X} = (\mathcal{P}_-)_Y \leq (\text{Env}_X(A_{X/X}))_Y \leq (A_{X/X})_Y = A_{Y/X}.$$

The lemma follows. \square

Inspired by the lemma above, we give the following definition:

Definition 4.2. Let X be a normal variety and $f : Y \rightarrow X$ be a movable modification. The diminished positive part \mathcal{P}_- is an \mathbb{R} -Weil b -divisor over X :

$$\mathcal{P}_- = \mathcal{P}_\sigma(K_Y + E_f) + \text{Env}_X(K_X).$$

Lemma 4.3. Let $f : Y \rightarrow X$ be a movable modification, $g : Z \rightarrow Y$ be a \mathbb{Q} -factorial birational model and $\phi = f \circ g$. Then the trace of \mathcal{P}_- on Z is $P_\sigma(K_Z + E_\phi) - \phi^*K_X$. In particular, \mathcal{P}_- is independent of the choice of Y .

Proof. We only need to show that

$$P_\sigma(K_Z + E_\phi) = P_\sigma(g^*(K_Y + E_f)).$$

Since (Y, E_f) is dlt, we have $K_Z + E_\phi \geq g^*(K_Y + E_f)$. Hence,

$$P_\sigma(K_Z + E_\phi) \geq P_\sigma(g^*(K_Y + E_f)).$$

Now we have

$$P_\sigma(g^*(K_Y + E_f)) \leq P_\sigma(K_Z + E_\phi) \leq K_Z + E_\phi.$$

As we know that $K_Y + E_f \in \overline{\text{Mov}}(Y/X)$, by Proposition 2.8(3), $g^*(K_Y + E_f) - P_\sigma(g^*(K_Y + E_f))$ is g -exceptional, and so is $K_Z + E_\phi - P_\sigma(g^*(K_Y + E_f))$. We obtain that $P_\sigma(K_Z + E_\phi) - g^*(K_Y + E_f)$ must be g -exceptional. Notice that $P_\sigma(K_Z + E_\phi) - g^*(K_Y + E_f) \in \overline{\text{Mov}}(Z/Y)$. By Lemma 2.3, we have $P_\sigma(K_Z + E_\phi) \leq g^*(K_Y + E_f)$. Proposition 2.8(2) gives the desired inequality. \square

Remark 4.4. If we assume the termination of MMP for dlt pairs, then there exists a movable modification with $K_Y + E_f$ nef over X , which is called a dlt modification [OX12]. In this case, it is not hard to see that $\mathcal{P}_\sigma(K_Y + E_f)$ is a \mathbb{Q} -Cartier b -divisor determined by f .

Lemma 4.5. Let $\phi : Z \rightarrow X$ be a \mathbb{Q} -factorial birational model over X not necessarily factoring through a movable modification. Then $(\mathcal{P}_-)_Z \leq P_\sigma(K_Z + E_\phi) - \phi^*K_X$.

Proof. Let $f : Y \rightarrow X$ be a movable modification and W be a common resolution of Y and Z as below:

$$\begin{array}{ccc} & W & \\ s \swarrow & & \searrow t \\ Y & & Z \end{array}$$

We only need to show that $t_*P_\sigma(f^*(K_Y + E_f)) \leq P_\sigma(K_Z + E_\phi)$. It is clear that

$$t_*P_\sigma(f^*(K_Y + E_f)) \leq t_*(f^*(K_Y + E_f)) \leq t_*(K_W + E_{f \circ s}) = K_Z + E_\phi$$

and $t_*P_\sigma(f^*(K_Y + E_f)) \in \overline{\text{Mov}}(Z/X)$. The lemma now follows from Proposition 2.8(2). \square

Definition 4.6. Suppose that $(X, 0)$ has only isolated singularity at $0 \in X$. We define the **movable volume** $\text{vol}_{\text{mov}}(X) = -(\mathcal{P}_-)^n$.

Lemma 4.7. *We have the following inequality between the volumes:*

$$0 \leq \text{vol}_{\text{BdFF}}(X) \leq \text{vol}_{\text{mov}}(X) < +\infty.$$

Proof. According to [Kollár13, 1.12.1], there exists a log resolution $s : W \rightarrow X$ with an effective s -exceptional divisor H on W such that $-H$ is s -ample. We fix an integer m such that $K_W + E_s - mH$ and $-s^*K_X - mH$ are both s -ample. For any model $\phi : Z \rightarrow X$ dominating a movable model and factoring through s via $t : Z \rightarrow W$, we have

$$\begin{aligned} (\mathcal{P}_-)_Z &= P_\sigma(K_Z + E_\phi) - \phi^*K_X \\ &\geq P_\sigma(t^*(K_W + E_s)) + P_\sigma(-t^*(s^*K_X)) \\ &\geq P_\sigma(t^*(K_W + E_s - mH)) + P_\sigma(t^*(-s^*K_X - mH)) \\ &= t^*(K_W + E_s - mH) + t^*(-s^*K_X - mH) \\ &= t^*(A_{W/X} - 2mH). \end{aligned}$$

Thus, by Proposition 2.18,

$$\text{vol}_{\text{mov}}(X) = -(\mathcal{P}_-)^n \leq -(A_{W/X} - 2mH)^n < +\infty.$$

By the proof of Lemma 4.1, we have $\mathcal{P}_- \leq \text{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$. Hence,

$$\text{vol}_{\text{mov}}(X) = -(\mathcal{P}_-)^n \geq -(\text{Env}_{\mathcal{X}}(A_{\mathcal{X}/X}))^n = \text{vol}_{\text{BdFF}}(X) \geq 0.$$

\square

Theorem 4.8. $\text{vol}_{\text{mov}}(X) = 0$ if and only if X is \mathbb{Q} -Gorenstein and log canonical.

Proof. Suppose that $\text{vol}_{\text{mov}}(X) = -(\mathcal{P}_-)^n = 0$. By Proposition 2.18, we have that $\mathcal{P}_- = 0$. Let $f : Y \rightarrow X$ be a movable modification, $g : Z \rightarrow Y$ be any \mathbb{Q} -factorial

model over Y and $\phi = f \circ g$. Then $0 = (\mathcal{P}_-)_Y = A_{Y/X}$, hence $K_Y + E_f = f^*K_X$. In particular, f^*K_X is \mathbb{Q} -Cartier. For the trace on Z , we have

$$\begin{aligned} 0 &= (\mathcal{P}_-)_Z = P_\sigma(g^*(K_Y + E_f)) - \phi^*K_X \\ &= P_\sigma(g^*(f^*K_X)) + P_\sigma(-g^*(f^*K_X)) \\ &\leq g^*(f^*K_X) - g^*(f^*K_X) = 0, \end{aligned}$$

where the second row follows from Lemma 2.15. In particular, we obtain that $\text{Env}_X(-K_X) = \mathcal{P}_\sigma(-f^*K_X)$ is a \mathbb{Q} -Cartier b -divisor determined by f . Thus, $-f^*K_X$ is f -nef by Lemma 2.16. Similarly, since $P_\sigma(g^*(f^*K_X)) = g^*(f^*K_X)$, we have that f^*K_X is also f -nef. Thus, f^*K_X is f -numerically trivial and X is numerically \mathbb{Q} -Gorenstein. The inequality $0 = \mathcal{P}_- \leq A_{\mathcal{X}/X}$ yields that X is numerically log canonical. We may apply Proposition 3.5 to conclude the proof. The other direction is obvious. \square

5. FINITE ENDOMORPHISM

We briefly review the pullback of Weil b -divisors.

Let $\phi : Y \rightarrow X$ be a generically finite dominant morphism between normal varieties. Every divisorial valuation ν on Y induces a divisorial valuation $\phi_*\nu$ via the natural inclusion of the function field $\phi^* : k(X) \hookrightarrow k(Y)$ given by $\phi_*\nu(f) = \nu(f \circ \phi)$ [BdFF12, Lemma 1.13]. In the other word, suppose that F is a divisor on X' over X . Then there is a birational model Y' over Y such that ϕ lifts to a morphism $\phi' : Y' \rightarrow X'$. We may obtain such Y' by blowing up the indeterminacy of the rational map $Y \dashrightarrow X'$. If E is an irreducible component of $(\phi')^{-1}F$ such that $\phi'(E) = F$, then $\phi_*\nu_E$ is a scalar multiple of ν_F with the scalar $\nu_E(\phi'^*F)$.

Let W be an \mathbb{R} -Weil b -divisor on X . We define the pullback of W as the \mathbb{R} -Weil b -divisor ϕ^*W such that $\nu_E(\phi^*W) = (\phi_*\nu_E)(W)$.

For a normal variety Z , we denote K_Z the Weil b -divisor given by a canonical divisor on each model and E_Z the Weil b -divisor given by the reduced exceptional divisor over Z on each model.

Lemma 5.1. [BdFF12, Lemma 2.19 and 3.3] *Let $\phi : Y \rightarrow X$ be a finite dominant morphism between normal varieties. Then*

- (1) *If D is an \mathbb{R} -Weil divisor on X , then $\text{Env}_Y(\phi^*D) = \phi^* \text{Env}_X(D)$.*
- (2) *If \mathcal{W} is an \mathbb{R} -Weil b -divisor over X such that $\text{Env}_X(\mathcal{W})$ is well-defined, then $\text{Env}_Y(\phi^*\mathcal{W}) = \phi^* \text{Env}_X(\mathcal{W})$.*
- (3) *If E is a prime exceptional divisor over Y , then*

$$\nu_E(K_Y + E_Y) = \nu_E(\phi^*(K_X + E_X)).$$

- (4) *If ϕ is étale in codimension 1, then $K_Y + E_Y = \phi^*(K_X + E_X)$.*

Proof. Part (1) and (2) are the same as [BdFF12, Lemma 2.19].

Let E be an exceptional divisor over Y . Let X' be a smooth model over X such that the center of $\phi_*\nu_E$ on X' is a divisor F . Let Y' be a smooth model over Y such that E is a divisor on Y' and $\phi' : Y' \dashrightarrow X'$ is a morphism. We summarize this in the following diagram:

$$\begin{array}{ccc} E \subset Y' & \xrightarrow{\phi'} & X' \supset F \\ g \downarrow & & \downarrow f \\ Y & \xrightarrow{\phi} & X. \end{array}$$

We have $\phi_*\nu_E = b\nu_F$ where $b = \nu_E(\phi'^*F)$. Hence the ramification order of ϕ' at the generic point of E is $b - 1$. We get

$$\nu_E(K_{Y'}) = \nu_E(\phi'^*K_{X'}) + b - 1 = b\nu_F(K_{X'}) + b - 1.$$

Thus

$$\nu_E(K_Y + E_Y) = \nu_E(K_{Y'}) + 1 = b\nu_F(K_{X'}) + b = \phi_*\nu_E(K_X + E_X).$$

Part (3) follows.

If ϕ is étale in codimension 1, then $K_Y = \phi^*K_X$. Hence, $K_Y + E_Y$ and $\phi^*(K_X + E_X)$ also coincide on non-exceptional valuations. \square

Lemma 5.2. *Let $f : Y \rightarrow X$ be a movable modification. Then*

$$\text{Env}_X(K_X + E_X) = \mathcal{P}_\sigma(K_Y + E_f).$$

In particular, $\mathcal{P}_- = \text{Env}_X(K_X + E_X) + \text{Env}_X(-K_X)$.

Proof. The proof is similar to Lemma 4.1. Let $g : Z \rightarrow X$ be any \mathbb{Q} -factorial model factoring through f via $\phi : Z \rightarrow Y$. By Lemma 4.3, the trace on Z satisfies:

$$(\mathcal{P}_\sigma(K_Y + E_f))_Z = \mathcal{P}_\sigma(K_Z + E_g) \leq K_Z + E_g.$$

Hence, $(\text{Env}_X(K_X + E_X))_Z \leq (\mathcal{P}_\sigma(K_Y + E_f))_Z$. On the other hand, $\mathcal{P}_\sigma(K_Y + E_f) \leq K_X + E_X$. By the definition of nef envelope, we have $\text{Env}_X(K_X + E_X) \geq \mathcal{P}_\sigma(K_Y + E_f)$. \square

We are ready to prove the main theorem.

Theorem 5.3. *Let $\phi : (Y, 0) \rightarrow (X, 0)$ be a finite morphism of degree d between normal isolated singularities such that ϕ is étale in codimension 1. Then*

- (1) $\mathcal{P}_{-,Y} = \phi^*\mathcal{P}_{-,X}$.
- (2) $\text{vol}_{\text{mov}}(Y) = d \text{vol}_{\text{mov}}(X)$.

Proof. By Lemma 5.1 and 5.2,

$$\begin{aligned}
\mathcal{P}_{-,Y} &= \text{Env}_Y(K_Y + E_Y) + \text{Env}_Y(-K_Y) \\
&= \text{Env}_Y(\phi^*(K_X + E_X)) + \text{Env}_Y(-\phi^*K_X) \\
&= \phi^* \text{Env}_X(K_X + E_X) + \phi^* \text{Env}_X(-K_X) \\
&= \phi^* \mathcal{P}_{-,X}.
\end{aligned}$$

Part (2) now follows from Lemma 2.18. \square

Theorem 5.4. *If $\phi : (X, 0) \rightarrow (X, 0)$ is a finite endomorphism of degree $d \geq 2$, then $\text{vol}_{\text{mov}}(X) = 0$.*

Proof. By Theorem 5.3, $\text{vol}_{\text{mov}}(X) = d \text{vol}_{\text{mov}}(X)$. But $\text{vol}_{\text{mov}}(X)$ is a nonnegative finite number. There must be $\text{vol}_{\text{mov}}(X) = 0$. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109, USA
E-mail address: zyuchen@umich.edu